Off-shell Bethe ansatz equations and N-point correlators in the SU(2) WZNW theory

1993 J. Phys. A: Math. Gen. 26 6981

(http://iopscience.iop.org/0305-4470/26/23/037)

J. Phys. A: Math. Gen. 26 (1993) 6981–6990. Printed in the UK

Off-shell Bethe ansatz equations and N-point correlators in the $SU(2)$ WZNW theory

H M Babujian†

Physikalisches Institut der Universität Bonn, Nussallee 12, D-53115 Bonn, Federal Republic of Germany

Received 29 July 1993

Abstract. We prove that the wavevectors of the off-shell Bethe ansatz equation for the inhomogeneous $SU(2)$ lattice vertex model render in the quasiclassical limit the solutions of the Knizhnik–Zamolodchikov equation.

1. Introduction

There exists a rather large class of integrable vertex models in 2D statistical mechanics, and among them many are gapless. The long-range behaviour of these gapless models is described by conformal field theories (CFT). The finite size resolution of the Bethe ansatz equations provides the values of the central charge and the conformal dimensions for the CFTs corresponding to integrable vertex models [1]. It is well known, that the Yang–Baxter equation plays the central role in constructing an integrable vertex model in 2D statistical mechanics. With each simple Lie algebra is associated a solution of the Yang–Baxter equation and with it is given an integrable vertex model. Most results so far obtained are related to homogeneous vertex models. We will consider in this note, instead, an inhomogeneous vertex model. There is, in the case of inhomogeneous models, associated with each vertex besides the spectral parameter λ also a disorder parameter z (one for each side). The vertex weight matrix R depends on $\lambda - z$. The transfer-matrix of the vertex model thus depends on disorder parameter z_i, $i = 1, 2, \ldots, N$. Transfer-matrices with different values of spectral parameter λ commute with each other [2–4], which means that the models are integrable (section 2). The purpose of this article is to investigate the connection between the integrable inhomogeneous vertex model and conformal field theory. The main ingredient of our approach will be the wavevector $\Phi(\lambda_1\lambda_2, \ldots, \lambda_n)$ of the algebraic Bethe ansatz satisfying, by construction, an equation of the form [5]

$$T(\lambda)\Phi(\lambda_1\lambda_2\ldots\lambda_n) = \Lambda(\lambda, \lambda_1, \lambda_2\ldots\lambda_n)]\Phi(\lambda_1, \ldots, \lambda_n) - \sum_{\alpha=1}^{n} \frac{F_\alpha \Phi_\alpha}{\lambda - \lambda_\alpha}. \tag{1.1}$$

Here $T(\lambda)$ denotes the transfer matrix of the vertex model acting on an N-fold tensor product of $SU(2)$ representation spaces.

Where $\Phi_\alpha = \Phi(\lambda_1, \ldots\lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1}, \ldots, \lambda_n)$, i.e. in $\Phi_\alpha \lambda_\alpha$ is replaced by λ. $F_\alpha(\lambda_1, \ldots\lambda_n)$ and $\Lambda(\lambda, \lambda_1, \ldots\lambda_n)$ are c numbers (section 2). The vanishing of the so-called

† Permanent address: Yerevan Physics Institute, Alikhanian Brothers 2, Yerevan, 375036 Armenia.

0305-4470/93/236981+10$07.50

'unwanted terms' (the last term on the RHS of (1.1)) is enforced in the usual procedure of the Bethe ansatz by choosing the parameters $\lambda_1, \dots, \lambda_n$ s.t. the functions F_α vanish. Φ then becomes an eigenvector of the transfer matrix with eigenvalue $\Lambda(\lambda, \lambda_1 \dots \lambda_n)$. We will, however, not impose these 'mass shell' conditions. For us the 'unwanted' terms are wanted. We call (1.1) the off-shell-Bethe-ansatz equation (OSBAE). Note that all the objects in the OSBAE (1.1) depend on the disorder parameters $z_1, \dots, z_N$. Our main purpose in this article is to uncover a neat relationship between the wavevectors satisfying the OSBAE (1.1) and vector-valued solutions of the Knizhnik–Zamolodchikov (KZ) equation. The latter linear differential equation is of the form [6]

$$\kappa \frac{\mathrm{d}\psi}{\mathrm{d}z_j} = \sum_{i \neq j}^{N} \frac{t_j^a t_i^a}{z_j - z_i} \psi. \tag{1.2}$$

The variables $z_1, \dots, z_N$ will be related to the disorder parameters of the Bethe ansatz. $\psi(z_1, \dots, z_N)$ is a vector in the tensor product of spaces $V^{(1)} \otimes V^{(2)} \otimes \dots V^{(N)}$, where $V^{(i)} i = 1, \dots, N$ are representation spaces of the simple algebra g. The $t_i^a (a = 1, 2, \dots, \dim g)$ represent the Hermitian generators of the algebra g and act non-trivially on $V^{(i)}$, $\kappa = 1/2(C_v + K)$ and $\delta^{ab} C_V = f^{acd} f^{bcd}$ (f^{abc} denoting the structure constants of the algebra g). K is the central charge of the Kac–Moody algebra. In this article we consider equation (1.2) only for the group $SU(2)$. The starting point of our work is the Yang–Baxter equation. We construct inhomogeneous vertex models with the disorder parameters $\{z_i\}$ using $SU(2)$ invariant rational solutions of the Yang–Baxter equation (section 2). The *technique of the algebraic Bethe ansatz will allow us to find vectors $\Phi(\lambda_1, \dots, \lambda_n)$ satisfying* (1.1) (section 2). The solution of the Yang–Baxter equation and OSBAE depend on a parameter η (Planck-type constant). Section 3 is devoted to a discussion of the quasiclassical limit $\eta \to 0$ of the OSBAE, which is identified with a spin wave problem treated by Gaudin [11] some time ago. In section 4 we construct from $\Phi(\lambda_1, \dots, \lambda_n)$ a solution of the KZ equation. This article is a revised version of [12], which was circulated three years ago. The identification of the Gaudin spin problem with the quasiclassical limit of the general Bethe ansatz problem is a new result.

2. Inhomogeneous vertex model

Let us consider a two-dimensional $M \times N$ lattice with $N + 1$, in general different, types of spin variables placed in the following manner inhomogeneously on the links of the lattice; on all horizontal links are spin variables σ taking values $\pm\frac{1}{2}$. The variables in the jth column $j = 1, 2, \dots N$ take values of an $SU(2)$ representation with spin s_j. The interaction takes place only between spins located on neighbouring links and is described by the vertex weight matrix $R_{i_1 i_2}^{j_1 j_2}(\lambda - z)$; λ here is the usual spectral parameter, z is a local disorder parameter associated with the particular bond. Cyclic boundary conditions are imposed. We use the $SU(2)$ invariant solution of the Yang–Baxter equation [7]

$${}_\sigma R^{12}(\lambda - \mu)_s R^{13}(\lambda - z)_s R^{23}(\mu - z) = {}_s R^{23}(\mu - z)_s R^{13}(\lambda - z)_\sigma R^{12}(\lambda - \mu) \tag{2.1}$$

where ${}_\sigma R^{12}(\lambda)$ is the vertex weight matrix of the XXX-model with spin $\frac{1}{2}$ [8]:

$${}_\sigma R^{12}(\lambda) = I^1 \otimes I^2 + \frac{2\eta}{\eta - 2\lambda} \sigma_1 \otimes \sigma_2 \tag{2.2}$$

and

$$
{}_sR^{12}(\lambda - z) = I^1 \otimes I^2 + \frac{2\eta}{\eta - 2(\lambda - z)} \sigma_1 \otimes S_2 \tag{2.3}
$$

$\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are Pauli matrices, $\boldsymbol{S} = (S^1, S^2, S^3)$ denotes an operator of arbitrary spin, I^1, I^2 are unit operators in the respective representation spaces (corresponding to spin $\frac{1}{2}$ and s, respectively). The solution (2.3) has been used in connection with the Kondo problem [9] and also in the exact solution of the XXX-model with arbitrary spin [7]. The parameter η in the R-matrices (2.2), (2.3), supplies the quasiclassical expansion

$$
R^{12}(\lambda, \eta)|_{\eta=0} = I^1 \otimes I^2. \tag{2.4}
$$

The monodromy operator and transfer-matrix are given in terms of the R-matrices (2.3) by

$$
\begin{aligned}
J(\lambda, z) &= R^{ON}(\lambda - z_N) R^{ON-1}(\lambda - z_{N-1}) \dots R^{01}(\lambda - z_1) \\
T(\lambda, z) &= \mathrm{tr}_0 J(\lambda, z).
\end{aligned} \tag{2.5}
$$

In (2.5) the trace is taken in the horizontal two-dimensional space 0 and

$$
R^{0k}(\lambda - z_k) = I^0 \otimes I^k + \frac{2\eta}{\eta - 2(\lambda - Z_k)} \sigma_0 \otimes S_k \tag{2.6}
$$

operators S_k act in the vertical space $V^{(k)}$ and $(S_k)^2 = s_k(s_{k+1})$. One infers from (2.1) for the monodromy operator $J(\lambda, z)$ the relation

$$
{}_\sigma R^{12}(\lambda - \mu)(J^1(\lambda, z) \otimes J^2(\mu, z)) = (J^2(\mu, z) \otimes J^1(\lambda, z))_\sigma R^{12}(\lambda - \mu). \tag{2.7}
$$

Due to the fact that $T(\lambda, z) = tr_0 J(\lambda, z)$ we have a family of commuting transfer matrices as in the homogeneous case

$$
[T(\lambda, z), T(\mu, z)] = 0. \tag{2.8}
$$

It is possible to diagonalize the transfer matrix $T(\lambda, z)$ by the algebraic Bethe ansatz [5], in just the same way as in the homogeneous case [7, 9]. Here, we describe this diagonalization. Let the operators $A(\lambda, z)$, $B(\lambda, z)$, $C(\lambda, z)$, $D(\lambda, z)$ be given by

$$
J(\lambda, z) = \begin{pmatrix} A(\lambda, z) B(\lambda, z) \\ C(\lambda, z) D(\lambda, z) \end{pmatrix}. \tag{2.9}
$$

The matrix ${}_\sigma R(\lambda)$ in (2.7) can be represented as

$$
{}_\sigma \dot{R}(\lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} b(\lambda) &= \frac{\eta}{\eta - \lambda} \\ c(\lambda) &= \frac{\lambda}{\lambda - \eta}. \end{aligned}
$$

Equation (2.7) gives us the commutators between the elements of $J(\lambda, z)$. We write down the commutators we will use later on

$$
\begin{aligned}
&[A(\lambda, z), A(\mu, z)] = [D(\lambda, z), D(\mu, z)] = 0 \\
&[B(\lambda, z), B(\mu, z)] = [C(\lambda, z), C(\mu, z)] = 0
\end{aligned} \tag{2.10}
$$

$$
B(\lambda, z), A(\mu, z) = b(\lambda - \mu) B(\mu, z) A(\lambda, z) + c(\lambda - \mu) A(\mu, z) B(\lambda, z) \tag{2.11}
$$

$$
B(\mu, z) D(\lambda, z) = b(\lambda - \mu) B(\lambda, z) D(\mu, z) + c(\lambda - \mu) D(\lambda, z) B(\mu, z). \tag{2.12}
$$

Let us consider the highest weight factor

$$|\Omega\rangle = |s_1, s_1\rangle \otimes |s_2, s_2\rangle \otimes \ldots \otimes |s_N, s_N\rangle S_i^3|s_i, s_i\rangle = s_i|s_i, s_i\rangle \tag{2.13}$$

We have the following well-known relations for the elements of the monodromy matrix:

$$A(\lambda, z)|\Omega\rangle = \prod_{i=1}^{N}(1 + P_i(\lambda)s_i)|\Omega\rangle \tag{2.14}$$

$$D(\lambda, z)|\Omega\rangle = \prod_{i=1}^{N}(1 - P_i(\lambda)s_i)|\Omega\rangle \tag{2.15}$$

$$C(\lambda, z)|\Omega\rangle = 0. \tag{2.16}$$

In (2.15) and (2.16) $P_i(\lambda) = 2\eta/(\eta - 2(\lambda - z_i))$. The Bethe wavefunction is

$$\Phi(\lambda_1, \lambda_2, \ldots \lambda_n; \{z\}) = \prod_{\alpha=1}^{n} B(\lambda_\alpha, z)|\Omega\rangle. \tag{2.17}$$

Using (2.12)–(2.13) and (2.15)–(2.17), we find the action of the transfer matrix $T(\lambda, z) = A(\lambda, z) + D(\lambda, z)$ on the Bethe vector Φ:

$$T(\lambda, z)\Phi = \Lambda(\lambda, \lambda_1, \lambda_2, \ldots \lambda_n)\Phi - \sum_{\alpha=1}^{n} \frac{F_\alpha(\lambda, z)}{\lambda - \lambda_\alpha}\Phi_\alpha \tag{2.18}$$

where

$$\Lambda(\lambda, \lambda_1, \lambda_2, \ldots \lambda_n) = \prod_{i=1}^{N}(1 + P_i(\lambda)s_i)\prod_{\alpha=1}^{N}\frac{1}{c(\lambda_\alpha - \lambda)} + \prod_{i=1}^{N}(1 - P_i(\lambda)s_i)\prod_{\alpha=1}^{n}\frac{1}{c(\lambda - \lambda_\alpha)} \tag{2.19}$$

$$F_\alpha(\lambda, z) = \eta\prod_{i=1}^{N}(1 + P_i(\lambda)s_i)\prod_{\beta\neq\alpha}^{n}\frac{\lambda_\alpha - \lambda_\beta + \eta}{\lambda_\alpha - \lambda_\beta} - \eta\prod_{i=1}^{N}(1 - P_i(\lambda)s_i)\prod_{\beta\neq\alpha}^{n}\frac{\lambda_\alpha - \lambda_\beta - \eta}{\lambda_\alpha - \lambda_\beta} \tag{2.20}$$

$$\Phi_\alpha = \Phi(\lambda_1, \lambda_2, \ldots \lambda_{\alpha-1}, \lambda, \lambda_{\alpha+1}, \ldots \lambda_n). \tag{2.21}$$

In (2.22) λ_α is replaced by λ in Φ_α or in other words $B(\lambda_\alpha, z)$ is replaced by $B(\lambda, z)$. The next step in the traditional Bethe ansatz procedure would consist in imposing the vanishing of the so-called 'unwanted' term (the second term in (2.19)). This is achieved through an appropriate choice of the parameters $\lambda_1, \ldots, \lambda_n$ s.t. the functions F_α vanishes. One arrives at the true eigenvalue equation

$$T(\lambda, z)\Phi = \Lambda(\lambda, \lambda_1, \lambda_2, \ldots \lambda_n)\Phi. \tag{2.22}$$

The Bethe ansatz equations $F_\alpha = 0$ classify the eigenvectors and eigenvalues of the operator $T(\lambda, z)$. One can say that when the Bethe ansatz equations $F_\alpha = 0$ are satisfied, the Bethe wavefunction is on the mass shell. If the condition $F_\alpha = 0$ is not imposed, then, in general, we have the equation (2.19), and the Bethe wavefunction is off the mass shell. In this case we call equation (2.19) the off-shell Bethe ansatz equation (OSBAE).

3. The quasiclassical limit of OSBAE and non-local Gaudin Hamiltonians

By quasiclassical expansion one commonly understands the expansion of the vertex weight $R(\lambda, \eta)$ around some point η_0, such, that $R(\lambda, \eta_0) = I \otimes I$ [10]. In this case one can parameterize η so that $\eta_0 = 0$. In this section we calculate the quasiclassical limit of the OSBAE (2.19). Let us start from calculation of the $T(\lambda, z)$ which, besides the parameters λ, z_i, depends also on η. In the homogeneous case ($z_i = 0$) we have, for the rational solution of the Yang–Baxter equation, essentially the same structure in the limits $\eta \to 0$ and $\lambda \to \infty$. In the inhomogeneous case we have another situation, because the dependence on z_i is additive with λ.

For the power series expansion $T(\lambda, z)$ around the point $\eta = 0$

$$T(\lambda, z) = \sum_{k=0}^{\infty} \eta^k T_k(\lambda, z) \tag{3.1}$$

follows from (2.8) that

$$\sum_{k+m=l} [T_k(\lambda, z), T_m(\mu, z)] = 0 \qquad l, k, m = 0, 1, 2 \ldots \tag{3.2}$$

which means the existence of the integrable subsystem in the quasiclassical series (3.1). It is interesting to note that the operators $T_k(\lambda, z)$ in general do not commute with $T(\lambda, z)$. In order to find a quasiclassical expansion, we represent the R-matrices (2.6) in the following form:

$$R^{0i}(\lambda - z_i) = \begin{pmatrix} I + P_i(\lambda)S_i^3 & P_i(\lambda)S_i^- \\ P_i(\lambda)S_i^+ & 1 - P_i(\lambda)S_i^3 \end{pmatrix}. \tag{3.3}$$

At $\eta \ll 1$ we have

$$P_i(\lambda) = -\frac{\eta}{\lambda - z_i} - \frac{1}{2}\left(\frac{\eta}{\lambda - z_i}\right)^2. \tag{3.4}$$

Substituting the first term from (3.5) into (3.3), we find the classical r-matrix [10]. From (3.3) and (3.4) we obtain the quasiclassical expansion of the monodromy operators $J(\lambda, z)$ (with accuracy $0(\eta^3)$)

$$J(\lambda, z) = \prod_{i=1}^{N} \left\{ I_0 \otimes I_i + P_i(\lambda) \begin{pmatrix} S_i^3 & S_i^- \\ S_i^+ & -S_i^3 \end{pmatrix} \right\} \tag{3.5}$$

Simple calculations give us

$$A(\lambda, z) = I - \eta S^3(\lambda, z) + \eta^2 \sum_{i<i} \frac{S_i^3 S_j^3 + S_i^- S_j^+}{(\lambda - z_i)(\lambda - z_j)} + \frac{\eta^2}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} S^3(\lambda, z) + 0(\eta^3) \tag{3.6}$$

$$D(\lambda, z) = I - \eta S^3(\lambda, z) + \eta^2 \sum_{i<i} \frac{S_i^3 S_j^3 + S_i^+ S_j^-}{(\lambda - z_i)(\lambda - z_j)} - \frac{\eta^2}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} S^3(\lambda, z) + 0(\eta^3) \tag{3.7}$$

$$B(\lambda, z) = -\eta S^-(\lambda, z) + \eta^2 \sum_{i<i} \frac{S_i^3 S_j^- - S_i^- S_j^3}{(\lambda - z_i)(\lambda - z_j)} - \frac{\eta^2}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} S^-(\lambda, z) + 0(\eta^3) \tag{3.8}$$

$$C(\lambda, z) = -\eta S^+(\lambda, z) + \eta^2 \sum_{i<i} \frac{S_i^+ S_j^3 - S_i^3 S_j^+}{(\lambda - z_i)(\lambda - z_j)} - \frac{\eta^2}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda} S^+(\lambda, z) + 0(\eta^3) \tag{3.9}$$

In (3.6)–(3.9)

$$I = \prod_{l=1}^{N} \otimes I_l$$

and also we use the following notation

$$S^a(\lambda, z) = \sum_{i=1}^{N} \frac{S_i^a}{\lambda - z_i} \tag{3.10}$$

where $S^a = (S^1, S^2, S^3)$ and $S^{\pm}(\lambda, z) = S^1(\lambda, z) \pm \mathrm{i}S^2(\lambda, z)$. For the transfer matrix $T(\lambda, z) = A(\lambda, z) + D(\lambda z)$ we have

$$T(\lambda, z) = 2I + 2\eta^2 \sum_{j=1}^{N} \frac{H_j}{\lambda - z_j} \tag{3.11}$$

$$H_i = \sum_{i \neq j}^{N} \frac{S_j^a S_i^a}{z_j - z_i}. \tag{3.12}$$

One can see from (3.1) and (3.11) that $T_0(\lambda, z) = 2I$, $T_1(\lambda, z) = 0$ and the second term in (3.11) is equal to $T_2(\lambda, z)$. It is obvious that the operators H_j commute as a consequence of equation (3.2). Then we can calculate with (3.6)–(3.9) the quasiclassical limits of the objects (2.20)–(2.22):

$$\Phi(\lambda_1, \lambda_2, \ldots \lambda_n) = (-\eta)^n \prod_{\alpha=1}^{n} S^-(\lambda_\alpha, z)|\Omega\rangle + O(\eta^{n+1}) \tag{3.13}$$

$$\begin{aligned} \Lambda(\lambda, \lambda_1, \lambda_2, \ldots \lambda_n) = 2 + 2\eta^2 \Bigg\{ & \sum_{i\alpha} \frac{s_i}{(\lambda - z_i)(\lambda_\alpha - \lambda)} + \sum_{i \neq j} \frac{s_i s_j}{(\lambda - z_i)(\lambda - z_j)} \\ & + \sum_{\alpha \neq \beta} \frac{1}{(\lambda_\alpha - \lambda)(\lambda_\beta - \lambda)} \Bigg\} + O(\eta^3) \end{aligned} \tag{3.14}$$

$$F_\alpha = 2\eta^2 \left\{ \sum_{\alpha \neq \beta}^{n} \frac{1}{\lambda_\alpha - \lambda_\beta} - \sum_{i=1}^{N} \frac{S_i}{\lambda_\alpha - z_i} \right\} + O(\eta^3) \tag{3.15}$$

$$\Phi_\alpha = (-\eta)^n S^-(\lambda_1, z) \ldots S^-(\lambda_{\alpha-1}, z) S^-(\lambda, z) S^-(\lambda_{\alpha+1}, z) \ldots S^-(\lambda_n, z) \times |\Omega\rangle + O(\eta^{n+1}). \tag{3.16}$$

Substituting now (3.13)–(3.16) and (3.11) into (2.19) and combining the terms proportional to η^{n+1}, we obtain the first non-trivial consequence of OSBAE (2.19) in the quasiclassical limit

$$\sum_{j=1}^{N} \frac{H_j}{\lambda - z_j} \varphi = h\varphi - \sum_{\alpha=1}^{n} \frac{f_\alpha \varphi_\alpha}{\lambda - \lambda_\alpha} \tag{3.17}$$

Here the vectors φ and φ_α from (3.14) and (3.16) are proportional to η^2 in (3.14) and (3.15), respectively. Taking the residue in the pole $\lambda = z_j$ of (3.17) we have

$$H_j\varphi = h_j\varphi - \sum_{\alpha=1}^{n} \frac{f_\alpha S_j^-}{z_j - \lambda_\alpha}\varphi'_\alpha \tag{3.18}$$

where

$$h_j = \sum_{j\neq j}^{N} \frac{s_i s_j}{z_j - z_i} + \sum_{\alpha=1}^{n} \frac{s_i}{\lambda_\alpha - z_j} \tag{3.19}$$

$$f_\alpha = \sum_{\beta\neq\alpha}^{n} \frac{1}{\lambda_\alpha - \lambda_\beta} - \sum_{i=1}^{N} \frac{s_i}{\lambda_\alpha - z_i} \tag{3.20}$$

$$\varphi(\lambda, z) = \prod_{\alpha=1}^{n} S^-(\lambda_\alpha, z)|\Omega\rangle. \tag{3.21}$$

In (3.18) we define the vector φ'_α; $\varphi = S^-(\lambda_\alpha, z)\varphi'_\alpha$, i.e. in the vector φ'_α the operator $S^-(\lambda_\alpha, z)$ is omitted. Equations (3.18) and (3.19)–(3.21) reproduce Gaudin's results [11], which he found by considering the spectral problem for the set of operators H_j. From (3.18)–(3.21) it also follows that the Gaudin method is, in fact, a quasiclassical version of the algebraic Bethe ansatz. If in (3.18) we impose the condition $f_\alpha = 0$, then we obtain φ as an eigenvector of the operators H_j with eigenvalues h_j. Parameters $\lambda_1, \ldots \lambda_n$ have to be found from the quasiclassical Bethe ansatz equations $f_\alpha = 0$.

4. The integral representation for the N-point correlators in WZNW theory

Let us introduce the function $\chi(\lambda, z) = \chi(\lambda_1, \ldots, \lambda_n, z_1, \ldots, z_N)$ obeying the following differential relations [12]

$$\kappa\frac{\mathrm{d}\chi}{\mathrm{d}z_j} = h_j\chi \tag{4.1}$$

$$\kappa\frac{\mathrm{d}\chi}{\mathrm{d}\lambda_\alpha} = f_\alpha\chi \tag{4.2}$$

where $\kappa = 1/2(k+2)$ (in the case $SU(2)C_v = 2$). Taking into account (3.19) and (3.20) it is easy to verify that the zero curvature conditions are fulfilled $\mathrm{d}h_j/\mathrm{d}\lambda_\alpha = \mathrm{d}f_\alpha/\mathrm{d}z_j$.

The solution of (4.1) and (4.2) is

$$\chi(\lambda, z) = \prod_{i<j}^{N}(z_i - z_j)^{S_i S_j/\kappa} \prod_{\alpha<\beta}^{n}(\lambda_\alpha - \lambda_\beta)^{1/\kappa} \prod_{k\gamma}(z_k - \lambda_\gamma)^{-S_k/\kappa}. \tag{4.3}$$

We define vector function $\Psi(z_1 \ldots z_N)$ through multiple contour integrals as follows [12]

$$\Psi(z_1, \ldots z_N) = \oint \ldots \oint \chi(\lambda, z)\varphi(\lambda, z)\,\mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n. \tag{4.4}$$

The integrations are to be taken here over canonical cycles of the space $X = C^n - U(\lambda_\alpha = z_i)$ with coefficients from S^*_χ dual to the local system S_λ, that is defined by the monodromy group of the function $\chi(\lambda, z)$. It is now rather straightforward to show that the vector function $\Psi(z_1, \ldots, z_N)$ defined above is the solution of KZ equation (1.2). Substituting (4.5) into KZ equation (1.2) using OSBAE (3.18) and the defining relations for χ (4.1), we find

$$\oint \cdots \oint \left[\chi \frac{\mathrm{d}\varphi}{\mathrm{d}z_j} - \frac{1}{\kappa} \sum_{\alpha=1}^{n} \frac{S_j^- f_\alpha \chi \varphi'_\alpha}{z_j - \lambda_\alpha}\right] \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n = 0. \tag{4.5}$$

Taking into account (4.2) and the identity which follows directly from (3.21)

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z_j} = -\sum_{\alpha=1}^{n} \frac{\mathrm{d}}{\mathrm{d}\lambda_\alpha} \frac{S_j^- \varphi'_\alpha}{z_j - \lambda_\alpha} \tag{4.6}$$

one easily verifies that (4.6) boils down to the relation

$$\sum_{\alpha=1}^{n} \oint \cdots \oint \frac{\mathrm{d}}{\mathrm{d}\lambda_\alpha} \left[\frac{S_j^- \varphi'_\alpha \chi}{z_j - \lambda_\alpha}\right] \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n = 0.$$

It is evident that this equation is satisfied, because the contours are closed. Now we want to show that $\Psi(z_1 \ldots z_N)$ is a singlet with respect to the global $SU(2)$ [6]

$$S^3 \Psi = \sum_{i=1}^{N} S_i^3 \Psi = 0 \tag{4.7}$$

$$S^\pm \Psi = \sum_{i=1}^{N} S_i^\pm \Psi = 0. \tag{4.8}$$

Equation (4.7) can easily be verified if we take into account the relation

$$[S^3, S^-(\lambda, z)] = -S^-(\lambda, z). \tag{4.9}$$

We have

$$S^3 \Psi = \oint \cdots \oint \left[\sum_{i=1}^{N} s_i - n\right] \chi \varphi \, \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n = 0 \tag{4.10}$$

since we now impose the condition

$$n = \sum_{i=1}^{N} s_i.$$

In order to verify (4.8) we present the Bethe wavefunction φ in the correlation functions (4.5) in more explicit form. It is indeed given by a sum of integrals of the Aomoto–Gelfand type [13, 14]

$$\chi_{k_1, k_2 \ldots k_n}(z_1, \ldots z_N) = \oint \cdots \oint \frac{\chi(\lambda, z) \, \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_n}{(z_{k_1} - \lambda_1)(z_{k_2} - \lambda_2) \ldots (z_{k_n} - \lambda_n)}. \tag{4.11}$$

Where $k_\alpha = 1, 2, \ldots, N$. Aomoto had studied such integrals in connection with general hypergeometric functions. It follows from his work that we can take such cycles (contours) where the integral will be completely symmetric under the permutation of any λ_α. In this case it will not depend on $k_1, \ldots, k_n$ in (4.13), i.e. on repetitions of a given z in the denominator of the integrand. So, we denote these integrals $\chi_{q_1 \ldots q_n (z_1, \ldots z_N)}$; where the q_i are repetition numbers of given z in the denominator of (4.13). We can represent Ψ in the form

$$\Psi(z_1 \ldots z_N) = \oint \ldots \oint \chi(\lambda, z) \sum_{k_1=1}^{N} \frac{S_{k_1}^-}{(z_{k_1} - \lambda_1)} \ldots \sum_{k_n=1}^{N} \frac{S_{k_1}^-}{z_{k_n} - \lambda_n} |\Omega\rangle \mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_n$$
$$= n! \sum_{q_1+q_2 \ldots q_N = n} \chi_{q_1, \ldots q_N} \frac{(S_j^-)^{q_1}}{q_1!} \ldots \frac{(S_N^-)^{q_N}}{(q_N)!} |\Omega\rangle. \tag{4.12}$$

We have then

$$S^+ \varphi = \sum_{q_1+q_2 \ldots q_N = n} \sum_{j=1}^{N} \chi_{q_1 \ldots q_N} \frac{(S_1^-)^{q_1}}{(q_1)!} \ldots \frac{S_j^+ (S_j^-)^{q_1}}{(q_j)!} \ldots \frac{(S_N^-)^{q_N}}{(q_N)!} |\Omega\rangle. \tag{4.13}$$

Taking into account the commutation relation

$$[S^+, (S^-)^k] = k(S^-)^{k-1}(2S^3 + 1 - k)$$

we obtain

$$\sum_{q_1+q_2+\ldots q_N = n-1} \sum_{j=1}^{N} \chi_{q_1 \ldots q_j+1, \ldots q_N} (2s_j - q_j) \frac{(S_1^-)^{q_1}}{(q_1)!} \ldots \frac{(S_N^-)^{q_N}}{(q_N)!} |\Omega\rangle = 0 \tag{4.14}$$

or equivalently

$$\sum_{j=1}^{N} \chi_{q_1 \ldots q_j+1, \ldots q_N} (2s_j - q_j) = 0. \tag{4.15}$$

This coincides with one of Aomoto's identities for general hypergeometric functions [13]. The formula (4.14) establishes the relationship between our approach and that of other authors [15–18]. In order to calculate specific correlation functions, as in [6, 19], i.e. the correlation function of primary fields $\Phi_{m_i}^{S_i}(z_i), m_i = -s_i \ldots s_i$

$$\langle \Phi_{m_1}^{s_1}(z_1) \Phi_{m_2}^{s_2}(z_r) \ldots \Phi_{m_N}^{s_N}(z_N) \rangle \tag{4.16}$$

it is necessary to multiply $\Psi(z_1, \ldots, z_N)$ from the left-hand side by the vector

$$\langle m_1, s_1| \otimes \langle m_2, |s_2| \otimes \ldots \otimes \langle m_N, s_N|$$

where $\langle m_i, s_i|$ is defined as

$$\langle m_i, s_i| S_i^3 = m_i \langle m_i, s_i|$$

where we have the relation

$$\langle \Phi_{m_1}^{S_1}(z_1) \Phi_{m_2}^{S_2}(z_2) \ldots \Phi_{m_M}^{S_n}(z_N) \rangle = \langle m_1, s_1| \otimes \langle m_2, s_2| \otimes \ldots \otimes \langle m_N, S_N| \Psi(z_1, z_2 \ldots z_N). \tag{4.17}$$

For full conformity with the WZNW theory in equations (4.1)–(4.2) we take $\kappa = 1/2(k+2)$. However, it is clear, that our construction allows us to work with arbitrary κ. So the inhomogeneous vertex model with transfer matrix $T(\lambda, z)$ and OSBAE generate the correlators of the WZNW theory.

5. Conclusions and speculation

Many connections have been established in past years relating integrable spin models and 2D CFT. The main result of this paper consists in adding another link in this direction: the solutions of the rational $SU(2)$ KZ equation are identified modulo a scalar integrating factor with a Bethe wavevector of the algebraic Bethe ansatz for an inhomogeneous vertex model in quasiclassical limit. It has to be stressed that the connection exists between the KZ equation and the off-shell Bethe ansatz equation. It should be noted that the quasiclassical expansion can be reinterpreted as high-temperature expansion of a lattice vertex model, because the leading term of this expansion corresponds to maximal entropy of the lattice vertex model.

We find it rather likely in considering the structure of OSBAE that general Bethe wave vectors (beyond the quasiclassical limit) will give the solutions of quantum KZ equations of rational type. Related results for the trigonometric quantum KZ equations are due to Matsuo [20] and Reshetikhin [21].

Acknowledgments

The author thanks A Sedrakyan, R Pogosian, and specially to R Flume, for discussions and comments. The author thanks the members of the Physikalisches Institut, where this work has been finished, for the hospitality extended. This work was partially supported by the grant 211-5291 YPI of the German Bundesministerium für Forschung und Technologie.

References

[1] de Vega H J 1989 *Int. J. Mod. Phys.* A **4** 2371
[2] Baxter R J 1982 *Exactly Solved Models in Statistical Mechanics* (London: Academic)
[3] Kulisch and Reshitikhin N Yu 1983 *J. Phys. A: Math. Gen.* **16** L591
[4] de Vega H J 1984 *Nucl. Phys.* B **240** 495
[5] Faddeev L D, Sklyanin E K and Takhtajan L A 1979 *Theor. Mat. Fis.* **40** 194
[6] Knizhnik V G and Zamolodchikov A B 1984 *Nucl. Phys.* B **247** 83
[7] Babujian H M 1983 *Nucl. Phys.* B **214** 317
[8] Baxter R J 1972 *Ann. Phys.* **70** 323
[9] Fateev V A and Wiegmann P B 1981 *Phys. Rev. Lett.* **46** 1955
[10] Belavin A A and Drinfeld V G 1982 *Funct. Anal. Applic.* **16** 159
[11] Gaudin M 1976 *J. Physique* **37** 1087
[12] Babujian H M 1990 *Yerevan Physics Institute Preprint* YERPHI-1261 (47)-90
[13] Aomoto K 1987 *J. Math. Soc. Japan* **39** 191
[14] Vasilev V A, Gelfand I M and Zelevinskii A V 1987 *Funct. Anal. Priloz.* **21** 19
[15] Christe P and Flume R 1987 *Nucl. Phys.* B **282** 466
Christe P 1986 *Thesis* Bonn University
[16] Date E, Jimbo M, Matsuo A and Miwa T 1990 *Int. J. Mod. Phys.* **4** 1049
[17] Schechtman V V and Varchenko A N 1990 *Lett. Math. Phys.* **20** 279; 1991 *Inven. Math.* **106** 139
[18] Matsuo A 1990 *Commun. Math. Phys.* **134** 65
[19] Zamalodchikov A B and Fateev V A 1986 *Sov. J. Nucl. Phys.* V **43** 1031
[20] Matsuo A 1993 *Commun. Math. Phys.* **151** 263
[21] Reshetikhin N 1991 *Lett. Math. Phys.* **26** 153